Center Manifold and Exponentially-Bounded Solutions of a System of Parabolic Equations

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Abstract

In this paper we study the existence of exponentially-bounded solutions of the following non-linear system of parabolic equations with homogeneous Neumann boundary conditions

$$\begin{aligned} u_t &= D\Delta u + f(t,u), \quad t \geq 0, \quad u \in \Re^n, \\ \frac{\partial u}{\partial \eta} &= 0, \quad \text{on} \quad \partial\Omega \end{aligned}$$

where $f \in C^1(\Re \times \Re^n)$, $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ is a diagonal matrix with $d_i > 0, \quad i = 1, 2, \ldots, n$ and Ω is a bounded domain in $\Re^N$. Under some conditions we prove the existence of a continuous manifold such that any solution with initial condition in this manifold is exponentially bounded.

Key words. system of parabolic equations, exponentially bounded solutions, center manifold.

AMS(MOS) subject classifications. primary: 34G10; secondary: 35B40.

Running Title: EXPONENTIALLY BOUNDED SOLUTIONS FOR PARABOLIC EQS.

1 Introduction

In this paper we shall study the existence of exponentially bounded solutions for the following system of parabolic equations with homogeneous Neumann boundary conditions

$$
\begin{aligned}
u_t &= D\Delta u + f(t,u), \quad t \geq 0, \quad u \in \Re^n, && (1)\\
\frac{\partial u}{\partial \eta} &= 0, \quad \text{on} \quad \partial\Omega && (2)
\end{aligned}
$$

where $f \in C^1(\Re \times \Re^n)$, $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ is a diagonal matrix with $d_i > 0, \quad i = 1, 2, \ldots, n$ and Ω is a bounded domain in $\Re^N$,where N is a non-negative integer.

We shall assume the following hypothesis:

H1) There exists $L_f > 0$ such that

$$|f(t,0)| \leq L_f, \quad \forall t \in \Re. \tag{3}$$

H2) f is globally Lipschitz in u,i.e, there exists $L > 0$ such that

$$|f(t,u_1) - f(t,u_2)| < L|u_1 - u_2|, \quad \forall u_1, u_2 \in \Re^n. \tag{4}$$

The fact that the first eigenvalue λ_1 of $-\Delta$ with Neumann boundary conditions is equal zero, does not allow us to prove the existence of bounded solutions of (1). However, under above assumptions, roughly speaking we prove the following statement:

If $d = 2\min\{d_i : i = 1, 2, \ldots, n\}$, λ_2 is the second eigenvalue of $-\Delta$ and η is positive numbers such that $\eta < \lambda_2 d$, then there exists a continuous manifold $\mathcal{M} = \mathcal{M}(\eta, d, f)$ such that any solution u of (1) starting in $\mathcal{M}$ satisfies the following estimate

$$\sup_{t\in\Re} e^{-\eta|t|} \left\{ \sup_{x\in\Omega} \|u(t,x)\| \right\} < \infty.$$

Several mathematical models may be written as a reaction-diffusion system of the form (1), like a model of vibration of plates (see [1]) and a Lotka-Volterra system with diffusion (see [7]). Some notations for this work can be found in [4], [5], [2] and [6].

2 Notation and Preliminaries

In this section we shall choose the space where this problem will be set.

Let $X = L^2(\Omega) = L^2(\Omega, \Re)$ and consider the linear unbounded operator $A : D(A) \subset X \to X$ defined by $A\phi = -\Delta\phi$, where

$$D(A) = \{\phi \in H^2(\Omega, \Re) : \frac{\partial \phi}{\partial \eta} = 0 \text{ on } \partial\Omega\}. \tag{5}$$

Since this operator is sectorial, then the fractional power space X^α associated with A can be defined. That is to say: for $\alpha \geq 0$, $X^\alpha = D(A_1^\alpha)$ endowed with the graph norm

$$\|x\|_\alpha = \|A_1^\alpha x\|, \quad x \in X^\alpha \text{ and } A_1 = A + aI, \tag{6}$$

where $\mathrm{Re}\sigma(A_1) > 0$. The norm $\|\cdot\|$ does not depend on a (see D. Henry [3] pg 29).

Precisely we have the following situation: Let $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty$ be the eigenvalues of A each one with finite multiplicity γ_j equal to the dimension of the corresponding eigenspace. Therefore

a) there exists a complete orthonormal set $\{\phi_{j,k}\}$ of eigenvector of A.

b) for all $x \in D(A)$ we have

$$Ax = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} < x, \phi_{j,k} > \phi_{j,k} = \sum_{j=1}^{\infty} \lambda_j E_j x, \tag{7}$$

where $< \cdot, \cdot >$ is the inner product in X and

$$E_j x = \sum_{k=1}^{\gamma_j} < x, \phi_{j,k} > \phi_{j,k}. \tag{8}$$

So, $\{E_j\}$ is a family of orthogonal projections in X and $x = \sum_{j=1}^{\infty} E_j x, \quad x \in X$.

c) $-A$ generates an analytic semigroup $\{e^{-At}\}$ given by

$$e^{-At}x = E_1 x + \sum_{j=2}^{\infty} e^{-\lambda_j t} E_j x. \tag{9}$$

d) for $a > 0$

$$X^{\alpha} = D(A_1^{\alpha}) = \{x \in X : \sum_{j=1}^{\infty} (\lambda_j + a)^{2\alpha} \|E_j x\|^2 < \infty\},$$

and

$$A_1^{\alpha} x = \sum_{j=1}^{\infty} (\lambda_j + a)^{\alpha} E_j x. \tag{10}$$

Also, we shall use the following notation:

$$Z := L^2(\Omega, \Re^n) = X^n = X \times \cdots \times X \ ,$$

with the usual norm.

Now, we define the following operator

$$\mathcal{A}_D : D(\mathcal{A}_D) \subset Z \to Z, \quad \mathcal{A}_D\psi = -D\Delta\psi = DA\psi, \tag{11}$$

where

$$D(\mathcal{A}_D) = [D(A)]^n = \{\phi \in H^2(\Omega, \Re^n) : \frac{\partial \phi}{\partial \eta} = 0 \text{ on } \partial\Omega\}.$$

Therefore, $\mathcal{A}_D$ is a sectorial operator and the fractional power space Z^α associated with $\mathcal{A}_D$ is given by

$$Z^\alpha = D(\mathcal{A}_{D1}^\alpha) = X^\alpha \times \cdots \times X^\alpha = [X^\alpha]^n. \tag{12}$$

endowed with the graph norm

$$\|z\|_\alpha = \|\mathcal{A}_{D1}^\alpha z\|, \quad z \in Z^\alpha \text{ and } \mathcal{A}_{D1} = \mathcal{A}_D + aI, \tag{13}$$

where

$$a > 0, \quad \mathcal{A}_{D1}^\alpha z = \sum_{j=1}^{\infty} D^\alpha(\lambda_j + a)^\alpha P_j z, \qquad D^\alpha = \operatorname{diag}(d_1^\alpha, d_2^\alpha, \cdots, d_n^\alpha), \tag{14}$$

and $P_j = \operatorname{diag}(E_j, E_j, \cdots, E_j)$, $n \times n$ matrix.

The c_o-semigroup $\{e^{-\mathcal{A}_D t}\}_{t\geq 0}$ generated by $-\mathcal{A}_D$ is given as follow

$$e^{-\mathcal{A}_D t} z = P_1 z + \sum_{j=2}^{\infty} e^{-\lambda_j D t} P_j z, \quad z \in Z. \tag{15}$$

Clearly, $\{P_j\}$ is a family of orthogonal projections in Z, which is complete. Hence, for $z = (z_1, z_2, ..., z_n)^T \in Z^\alpha$ we have that

$$\begin{aligned} z = \sum_{j=1}^{\infty} P_j z \ , \|z\|^2 = \sum_{j=1}^{\infty} \|P_j z\|^2 \text{ and } \|z\|_\alpha^2 &= \sum_{j=1}^{\infty} \|P_j z\|_\alpha^2 \\ &= \sum_{j=1}^{\infty} \sum_{i=1}^{n} ((\lambda_j + a)d_i)^{2\alpha} \|E_j z_i\|^2 \ . \end{aligned}$$

Using the foregoing notations we shall prove the following propositon

Proposition 2.1 *Let* $\pi_0 = P_1$ *and* $\pi_s = I - P_1$. *Then for all* $z \in Z^\alpha$ *the following inequalities hold*

$$\|e^{-\mathcal{A}_\mathcal{D} t}\pi_0 z\|_\alpha \leq \|z\|_\alpha, \quad t \in \Re \tag{16}$$

$$\|e^{-\mathcal{A}_\mathcal{D} t}\pi_s z\|_\alpha \leq e^{-\lambda_2 dt}\|z\|_\alpha, \quad t \geq 0 \tag{17}$$

$$\|e^{-\mathcal{A}_\mathcal{D} t}\pi_s z\|_\alpha \leq M t^{-\alpha} e^{-\lambda_2 dt}\|z\|, \quad t > 0 \tag{18}$$

$$I_Z = \pi_0 + \pi_s, \quad Z = Z_c \oplus Z_s, \tag{19}$$

where $M = \sup_{t\geq 0}\left\{((\lambda_j + a)d_i t)^\alpha e^{-\lambda_j \frac{d_i}{2}t}, \quad i = 1, 2, \dots, n; j = 1, 2, \dots\right\}$, $2d = \min\{d_i : i = 1, ..., n\}$ *and* $Z_0 = \text{Range}(\pi_0) = \mathcal{R}(\pi_0)$, $\quad Z_s = \text{Range}(\pi_s) = \mathcal{R}(\pi_s)$.

Proof . From the above notation, for $z \in Z^\alpha$ we have that

$$\|e^{-\mathcal{A}_D t}\pi_0 z\|_\alpha^2 = \|\mathcal{A}_D^\alpha e^{-\mathcal{A}_D t}\pi_0 z\|^2 = \|D^\alpha a^\alpha E_1 z\|^2 \leq \|z\|_\alpha^2.$$

Therefore,

$$\|e^{-\mathcal{A}_\mathcal{D} t}\pi_0 z\|_\alpha \leq \|z\|_\alpha.$$

Next, we shall prove the second inequality,

$$\begin{aligned}
\|e^{-\mathcal{A}_D t}\pi_s z\|_\alpha^2 &= \sum_{j=1}^{\infty}\sum_{i=1}^{n}((\lambda_j + a)d_i)^{2\alpha} e^{-2\lambda_j d_i t}\|E_j z_i\|^2 \\
&\leq e^{-2\lambda_2 dt}\sum_{j=1}^{\infty}\sum_{i=1}^{n}((\lambda_j + a)d_i)^{2\alpha}\|E_j z_i\|^2 \\
&\leq \|z\|_\alpha^2 e^{-2\lambda_2 dt}
\end{aligned}$$

Therefore,

$$\|e^{-\mathcal{A}_D t}\pi_s z\|_\alpha \le \|z\|_\alpha e^{-\lambda_2 dt}, \quad t \ge 0.$$

Finally, we prove the last inequality

$$\begin{aligned}
\|e^{-\mathcal{A}_D t}\pi_s z\|_\alpha^2 &= \sum_{j=2}^{\infty}\sum_{i=1}^{n}((\lambda_j + a)d_i)^{2\alpha} e^{-2\lambda_j d_i t}\|E_j z_i\|^2 \\
&= \frac{1}{t^{2\alpha}}\sum_{j=2}^{\infty}\sum_{i=1}^{n}\{((\lambda_j + a)d_i t)^{\alpha} e^{-\lambda_j d_i t}\}^2\|E_j z_i\|^2 \\
&= \frac{1}{t^{2\alpha}}\sum_{j=2}^{\infty}\sum_{i=1}^{n}\{((\lambda_j + a)d_i t)^{\alpha} e^{-\lambda_j \frac{d_i}{2} t}\}^2\{e^{-\lambda_j \frac{d_i}{2} t}\}^2\|E_j z_i\|^2 \\
&\le \frac{1}{t^{2\alpha}}\sum_{j=2}^{\infty}\sum_{i=1}^{n} M^2 e^{-\lambda_j d_i t}\|E_j z_i\|^2 \le \frac{1}{t^{2\alpha}} M^2 e^{-2\lambda_2 d}\|z\|^2.
\end{aligned}$$

Therefore,

$$\|e^{-\mathcal{A}_D t}\pi_s z\|_\alpha \le M t^{-\alpha}\|z\| e^{-d\lambda_2 t}, \quad t > 0.$$

$\square$

From Theorem 1.6.1 (pp. 39-40) in D.Henry [3] it follows for $0 < \alpha < 1$ that the following inclusion is continuous

$$Z^\alpha \quad \subset \quad L^p(\Omega, \Re^n), \; p \ge 2. \tag{20}$$

Hence, there exists $R > 0$ such that

$$\|z\| \quad \le \quad R\|z\|_\alpha, \; z \in Z^\alpha. \tag{21}$$

Now, the system (1)-(2) can be written in an abstract way on Z as follows:

$$z' = -\mathcal{A}_D z + f^e(t, z), \quad z(t_0) = z_0, \quad t \ge t_0, \tag{22}$$

where $f^e : \Re \times Z^\alpha \to Z$ is given by:

$$f^e(t,z)(x) = f(t,z(x)), \quad x \in \Omega. \tag{23}$$

To show that equation (22) is welll posed in Z^α we need the following proposition.

Proposition 2.2 *The function f^e given in (23) satisfies the estimation*

$$\|f^e(t,z_1) - f^e(t,z_2)\| \leq LR\|z_1 - z_2\|_\alpha, \; t \in \Re, z_1, z_2 \in Z^\alpha \;, \tag{24}$$

where L and R are as in (4) and (21), respectively. Also,

$$\|f^e(t,0)\| \leq \mu(\Omega)L_f, \; t \geq 0, \tag{25}$$

where $\mu(\Omega)$ is the Lebesgue measure of Ω.

Proof .

$$\begin{aligned} \|f^e(t,z_1) - f^e(t,z_2)\| &= \left\{ \int_\Omega |f(t,z_1(x)) - f(t,z_2(x))|^2 dx \right\}^{\frac{1}{2}} \\ &\leq L\|z_1 - z_2\| \\ &\leq LR\|z_1 - z_2\|_\alpha, \; t \in \Re, z_1, z_2 \in Z^\alpha. \end{aligned}$$

On the other hand, (3) implies (25). □

3 Main Theorem

From the proposition 2.2 and Theorem 7.1.4 in D.Henry ([3]), for all $t \geq t_0$, we have that a continuous function $z(\cdot) : (t_0, \infty) \to Z^\alpha$ is solution of the integral equation

$$z(t) = e^{-\mathcal{A}_D(t-t_0)}z_0 + \int_{t_0}^{t} e^{-\mathcal{A}_D(t-s)} f^e(s, z(s))ds, \quad t \geq t_0 \tag{26}$$

if and only if $z(\cdot)$ is a solution of (22).

Now, for all $\eta \geq 0$, we denote by Z^α_η the Banach space

$$Z^\alpha_\eta = \{z \in C(\Re, Z^\alpha) : \|z\|_{\alpha,\eta} := \sup_{t\in\Re} e^{-\eta|t|}\|z(t)\|_\alpha < \infty\} \ . \tag{27}$$

Theorem 3.1 *Suppose that f satisfies H_1 and H_2. Then for some a and d positive, and $0 < \eta < \lambda_2 d$, there exists a continuous manifold $\mathcal{M} = \mathcal{M}(a, d, f)$ such that any solution $z(t)$ of (22) with $z(0) \in \mathcal{M}$ satisfies the estimate*

$$\sup_{t\in\Re} e^{-\eta|t|}\|z(t)\|_\alpha < \infty \ ,$$

Moreover, there exists a globally Lipschitz function $\psi : \mathcal{R}(\pi_0) \to \mathcal{R}(\pi_s)$ such that

$$\mathcal{M} \ = \ \{z_0 + \psi(z_0) : z_0 \in \mathcal{R}(\pi_0)\} \ . \tag{28}$$

Before the proof of Theorem 3.1 we shall prove some preliminaries lemmas.

Lemma 3.1 *Let $z \in Z^\alpha_\eta$ and $\eta < \lambda_2 d$. Then z is a solution of (22) if and only if there exists $z_0 \in \mathcal{R}(\pi_0)$ such that*

$$\begin{aligned} z(t) \ &= \ e^{-\mathcal{A}_D t}z_0 + \int_0^t e^{-\mathcal{A}_D(t-s)}\pi_0 f^e(s, z(s))ds \\ &\quad + \int_{-\infty}^{t} e^{-\mathcal{A}_D(t-s)}\pi_s f^e(s, z(s))ds, \quad t \in \Re \ . \end{aligned} \tag{29}$$

Proof . Suppose that z is a solution of (22). Using the fact that $z(t) = \pi_0 z(t) + \pi_s z(t)$ and the variation of constants formula (26) , we obtain

$$\pi_0 z(t) \;=\; e^{-\mathcal{A}_D t}\pi_0 z(0) + \int_0^t e^{-\mathcal{A}_D(t-s)}\pi_0 f^e(s, z(s))ds, \quad t \in \Re,$$

and

$$\pi_s z(t) \;=\; e^{-\mathcal{A}_D(t-t_0)}\pi_s z(0) + \int_{t_0}^t e^{-\mathcal{A}_D(t-s)}\pi_s f^e(s, z(s))ds, \quad t \geq t_0. \tag{30}$$

Using (17), we obtain

$$\|e^{-\mathcal{A}_\mathcal{D}(t-t_0)}\pi_s z(t_0)\|_\alpha \;\leq\; e^{-\lambda_2 d(t-t_0)}\|z(t_0)\|_\alpha.$$

Therefore,

$$\|e^{-\mathcal{A}_\mathcal{D}(t-t_0)}\pi_s z(t_0)\|_\alpha \;\leq\; e^{-\lambda_2 d(t-t_0)}e^{\eta|t_0|}\|z\|_{\alpha,\eta}.$$

Since $\eta < \lambda_2 d$, then right side of this inequality goes to zero as t_0 goes to $-\infty$.

Now, from (18) and Proposition 2.1 the following chain of inequalities hold

$$\begin{aligned}
\|\int_{-\infty}^t e^{-\mathcal{A}_\mathcal{D}(t-s)}\pi_s f^e(s, z(s))ds\|_\alpha \;&\leq\; \int_{-\infty}^t M(t-s)^{-\alpha}e^{-\lambda_2 d(t-s)}\|f^e(s, z(s))\|ds \\
&\leq\; MLR\int_{-\infty}^t (t-s)^{-\alpha}e^{-\lambda_2 d(t-s)}\|z(s)\|_\alpha ds \\
&+\; ML_f\mu(\Omega)\int_{-\infty}^t (t-s)^{-\alpha}e^{-\lambda_2 d(t-s)}ds \\
&\leq\; MLR\|z\|_{\alpha,\eta}\int_{-\infty}^t (t-s)^{-\alpha}e^{-\lambda_2 d(t-s)}e^{\eta|s|}ds \\
&+\; ML_f\mu(\Omega)\int_{-\infty}^t (t-s)^{-\alpha}e^{-\lambda_2 d(t-s)}ds \quad .
\end{aligned}$$

We now pay attention to the integrals $\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-\lambda_2 d(t-s)}e^{\eta|s|}ds$ and $\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-\lambda_2 d(t-s)}ds$. If $t<0$, then through the change of variables $t-s=u$ and $\lambda_2 du = v$ we can obtain that

$$\begin{aligned}\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-\lambda_2 d(t-s)}e^{\eta|s|}ds &\leq e^{-\eta t}(\lambda_2 d)^{1-\alpha}\Gamma(1-\alpha)\\ &= e^{\eta|t|}(\lambda_2 d)^{1-\alpha}\Gamma(1-\alpha)\ .\end{aligned}$$

The previuos changes of variables also allow us to show that if $t>0$, then

$$\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-\lambda_2 d(t-s)}e^{\eta|s|}ds \leq 2e^{\eta|t|}(\lambda_2 d)^{1-\alpha}\Gamma(1-\alpha).$$

For the integral $\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-\lambda_2 d(t-s)}ds$, we obtain the following estimate

$$\begin{aligned}\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-\lambda_2 d(t-s)}ds &\leq \int_{-\infty}^{t}(t-s)^{-\alpha}e^{-\lambda_2 d(t-s)}e^{\eta|s|}ds\\ &\leq e^{\eta|t|}2(\lambda_2 d)^{1-\alpha}\Gamma(1-\alpha)\ .\end{aligned}$$

Going back to the expresion $\|\int_{-\infty}^{t}e^{-\mathcal{A}_{\mathcal{D}}(t-s)}\pi_s f^e(s,z(s))ds\|_\alpha$, we obtain that

$$\|\int_{-\infty}^{t}e^{-\mathcal{A}_{\mathcal{D}}(t-s)}\pi_s f^e(s,z(s))ds\|_\alpha \leq 2Me^{\eta|t|}(\lambda_2 d)^{1-\alpha}\Gamma(1-\alpha)[LR\|z\|_{\alpha,\eta}+L_f\mu(\Omega)].$$

Hence, the improper integral $\int_{-\infty}^{t}e^{-\mathcal{A}_D(t-s)}\pi_s f^e(s,z(s))ds$ exists and passing to the limit in (30), as t_0 goes to $-\infty$ produces

$$\pi_s z(t) = \int_{-\infty}^{t}e^{-\mathcal{A}_D(t-s)}\pi_s f^e(s,z(s))ds, \quad t\in\Re.$$

Therefore, letting $z_0=\pi_0 z(0)$ we get (29).

Conversely, suppose z is a solution of (29). Then

$$z(t) \;=\; e^{-\mathcal{A}_{\mathcal{D}}t}z_0+\int_0^t e^{-\mathcal{A}_D(t-s)}\pi_0 f^e(s,z(s))ds$$

$$
\begin{aligned}
&+ \int_0^t e^{-\mathcal{A}_D(t-s)}\pi_s f^e(s,z(s))ds + \int_{-\infty}^0 e^{-\mathcal{A}_D(t-s)}\pi_s f^e(s,z(s))ds \\
&= e^{-\mathcal{A}_\mathcal{D} t}[z_0 + \int_{-\infty}^0 e^{-\mathcal{A}_D(-s)}\pi_s f^e(s,z(s))ds] + \int_0^t e^{-\mathcal{A}_D(t-s)} f^e(s,z(s))ds \\
&= e^{-\mathcal{A}_\mathcal{D} t} z_0 + \int_0^t e^{-\mathcal{A}_D(t-s)} f^e(s,z(s))ds \quad ,
\end{aligned}
$$

where

$$
z(0) \;=\; z_0 + \int_{-\infty}^0 e^{-\mathcal{A}_D(-s)}\pi_s f^e(s,z(s))ds \quad . \tag{31}
$$

This concludes the proof of the lemma. □

Inspired in (29), the manifold $\mathcal{M}$ we are looking for is defined by

$$
\mathcal{M} \;=\; \{z(0) : z \in Z_\eta^\alpha \text{ and satisfies } (29)\} \, . \tag{32}
$$

A useful characterization of $\mathcal{M}$, to prove later Theorem 3.1, that follows from (31) is given by

$$
\mathcal{M} \;=\; \{z_0 + \pi_s z(0) : (z_0, z) \in \mathcal{R}(\pi_0) \times Z_\eta^\alpha, (z_0, z) \text{ satisfying } (29)\} \tag{33}
$$

We shall need some definitions and notations :

(a) For each $z_0 \in \mathcal{R}(\pi_0)$ we define the function $Sz_0 : \Re \to Z^\alpha$ by

$$
(Sz_0)(t) = e^{-\mathcal{A}_D t} z_0, \quad t \in \Re \quad .
$$

(b) For each function $z \in Z_\eta^\alpha$ we define the function $G : Z_\eta^\alpha \to Z_\eta^\alpha$ by

$$
\begin{aligned}
G(z)(t) \;=\; & \int_0^t e^{-\mathcal{A}_D(t-s)}\pi_0 f^e(s,z(s))ds \\
& + \int_{-\infty}^t e^{-\mathcal{A}_D(t-s)}\pi_s f^e(s,z(s))ds, \quad t \in \Re \quad .
\end{aligned}
$$

The fact that $G(Z^\alpha_\eta) \subset Z^\alpha_\eta$ follows from (24)-(25) and a similar computation done in proposition 2.1. Now, with the previous notations (29) can be written in the following equivalent form in Z^α_η

$$z = Sz_0 + G(z) \tag{34}$$

Lemma 3.2 *(a) For $0 < \eta < \lambda_2 d$, S is a bounded linear operator from $\mathcal{R}(\pi_0)$ in Z^α_η.*

(b) G is globally Lipschitz. Moreover, given z_1 and z_2 in Z^α_η we have

$$\|G(z_1) - G(z_2)\|_{\alpha,\eta} \quad \le \quad K\|z_1 - z_2\|_{\alpha,\eta}, \tag{35}$$

where

$$K \quad := \quad LR\left(2M(\lambda_2 d)^{1-\alpha}\Gamma(1-\alpha) + \frac{(\bar{d}a)^\alpha}{\eta}\right) , \tag{36}$$

and $\bar{d} = \|D\|$.

Proof . (a) Clearly S is linear and $\|Sz_0\|_{\alpha,\eta} \le \|z_0\|_\alpha$ for all $z_0 \in \mathcal{R}(\pi_0)$.

(b) Let z_1, z_2 be given in Z^α_η.

If $t > 0$, then

$\| \int_0^t e^{-\mathcal{A}_\mathcal{D}(t-s)}\pi_0[f^e(s, z_1(s)) - f^e(s, z_2(s))]\, ds\|_\alpha$

$$\le \int_0^t \|D^\alpha a^\alpha E_1[f^e(s, z_1(s)) - f^e(s, z_2(s))]\|\, ds \quad \le \quad \frac{(\bar{d}a)^\alpha LR}{\eta} e^{\eta|t|}\|z_1 - z_2\|_{\alpha,\eta}.$$

For $t < 0$, the same estimations hold. Hence, for all $t \in \Re$,

$$\| \int_0^t e^{-\mathcal{A}_\mathcal{D}(t-s)}\pi_0[f^e(s, z_1(s)) - f^e(s, z_2(s))]\, ds\|_\alpha$$

$$\leq \frac{(\bar{d}a)^{\alpha}LR}{\eta}e^{\eta|t|}\|z_1 - z_2\|_{\alpha,\eta} \quad . \tag{37}$$

Now, for all $t \in \Re$,

$$\| \int_{-\infty}^{t} e^{-\mathcal{A}_{\mathcal{D}}(t-s)}\pi_s[f^e(s, z_1(s)) - f^e(s, z_2(s))]\, ds\|_{\alpha}$$

$$\leq 2MLR(\lambda_2 d)^{1-\alpha}\Gamma(1-\alpha)e^{\eta|t|}\|z_1 - z_2\|_{\alpha,\eta} \quad . \tag{38}$$

Finally, from (37) and (38) we get

$$\begin{aligned}
\|G(z_1)(t) - G(z_2)(t)\|_{\alpha} &\leq \| \int_{0}^{t} e^{-\mathcal{A}_{\mathcal{D}}(t-s)}\pi_0[f^e(s, z_1(s)) - f^e(s, z_2(s))]\, ds\|_{\alpha} \\
&\leq \| \int_{-\infty}^{t} e^{-\mathcal{A}_{\mathcal{D}}(t-s)}\pi_s[f^e(s, z_1(s)) - f^e(s, z_2(s))]\, ds\|_{\alpha} \\
&\leq Ke^{\eta|t|}\|z_1 - z_2\|_{\alpha,\eta} \ ,
\end{aligned}$$

and this implies (35). □

Proof of Theorem 3.1 .

Let, a and d be given such that $K < 1$. Then $I - G : Z^{\alpha}_{\eta} \to Z^{\alpha}_{\eta}$ is a homeomorphism with inverse $\Psi : Z^{\alpha}_{\eta} \to Z^{\alpha}_{\eta}$. Ψ is also globally Lipschitz and for all $z_1, z_2 \in Z^{\alpha}_{\eta}$ we have

$$\|\Psi(z_1) - \Psi(z_2)\|_{\alpha,\eta} \ \leq \ (1-K)^{-1}\|z_1 - z_2\|_{\alpha,\eta}. \tag{39}$$

Therefore, the equation (34) has a unique solution given by

$$\begin{aligned}
z(t) &= (I-G)^{-1}(Sz_0)(t) \\
&= \Psi(Sz_0)(t), \quad t \in \Re.
\end{aligned}$$

Hence, from (33) we get that

$$\mathcal{M} \;=\; \{z_0 + \pi_s \Psi(Sz_0)(0) : z_0 \in \mathcal{R}(\pi_0)\}$$

and defining $\psi : \mathcal{R}(\pi_0) \to \mathcal{R}(\pi_s)$ by $\psi(z_0) = \pi_s \Psi(Sz_0)(0)$ we obtain (28). Next, we prove that ψ is globally Lipschitz. In fact, let z_0, z_1 be given in $\mathcal{R}(\pi_0)$. Then the estimation (39) implies

$$\begin{aligned} \sup_{t\in\Re} e^{-\eta|t|}\|\pi_s\Psi(Sz_0)(t) - \pi_s\Psi(Sz_1)(t)\|_\alpha &= \|\pi_s[\Psi(Sz_0) - \Psi(Sz_1)]\|_{\alpha,\eta} \\ &\leq (1-K)^{-1}\|\pi_s\|\|z_0 - z_1\|_\alpha , \end{aligned}$$

and, in particular for $t = 0$ we get

$$\|\psi(z_0) - \psi(z_1)\|_\alpha \;\leq\; (1-K)^{-1}\|\pi_s\|\|z_0 - z_1\|_\alpha .$$

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